Deriving the Lorentz Equations From the Constant Velocity or "Boost" Transformation

The so called "boost" transformation, where the resulting dimensions t' and x' are advancing, i.e. the spatial dimensions are "traveling" at a constant velocity relative to the original frame is

$$\Lambda_{\nu}^{\mu'} = \begin{bmatrix} \cosh(\phi) & -\sinh(\phi) & 0 & 0 \\ -\sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is analogous to a rotation where sin and cos are used instead.

With the dimension variables defined as

$$x^{\nu} : \begin{matrix} x^1 = t \\ x^2 = x \\ x^3 = y \\ x^4 = z \end{matrix} , x^{\mu} : \begin{matrix} x^1 = t' \\ x^2 = x' \\ x^3 = y' \\ x^4 = z' \end{matrix}$$

the transformation is

$$x^{\mu} = \Lambda_{\nu}^{\mu'} x^{\nu}$$

or

$$t' = t\cosh(\phi) - x\sinh(\phi)$$
$$x' = -t\sinh(\phi) + x\cosh(\phi)$$

The velocity, ν is

$$v = \frac{x}{t} = \frac{\sinh(\phi)}{\cosh(\phi)} = \tanh(\phi) \Rightarrow \phi = \tanh^{-1} v = \text{the boost parameter}$$

To derive the Lorentz equations, we replace ϕ with $\tanh^{-1}\nu$. To make things easier, we find a more agreeable form of $\tanh^{-1}x$:

$$y=\tanh(x)\Rightarrow x=\tanh^{-1}y$$

$$y=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\frac{e^{2x}-1}{e^{2x}-1}\Rightarrow e^{2x}-1=y(e^{2x}+1)=ye^{2x}+y\Rightarrow$$

$$e^{2x}-1-ye^{2x}-y=0=e^{2x}(1-y)-1-y\Rightarrow e^{2x}(1-y)=(1+y)\Rightarrow$$

$$e^{2x}=\frac{1+y}{1-y}\Rightarrow 2x=\ln\left(\frac{1+y}{1-y}\right)\Rightarrow x=\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)=\tanh^{-1}y$$

Replacing ϕ with $\tanh^{-1}v$:

$$\cosh(\phi)=\cosh(\tanh^{-1}v)=\cosh\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)$$

$$\frac{e^{\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)}+e^{\left(-\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)}}{2}=\frac{e^{\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)}+\frac{1}{e^{\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)}}}{2}=\frac{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}+\frac{1}{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}}{2}=$$

$$\frac{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}\frac{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}+\frac{1}{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}}{2}=\frac{\frac{\left(\frac{1+v}{1-v}\right)+1}{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}}{2}=\frac{\frac{1+v}{1-v}+1}{2\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}=\frac{\frac{1+v}{1-v}+\frac{1-v}{1-v}}{2\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}=\frac{\frac{2}{1-v}}{2\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}=$$

$$\frac{\frac{1}{1-v}\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}{1\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}=\frac{\frac{1}{1-v}\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}{\left(\frac{1+v}{1-v}\right)}=\frac{\frac{1}{1-v}\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}(1-v)}{1+v}=\frac{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}{1+v}=\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}\frac{1}{1+v}=$$

$$\left(\frac{1+v}{1-v}\left(\frac{1}{1+v}\right)\left(\frac{1}{1+v}\right)\right)^{\frac{1}{2}}=\left(\frac{1}{1-v}\left(\frac{1}{1+v}\right)\right)^{\frac{1}{2}}=\left(\frac{1}{1-v^{2}}\right)^{\frac{1}{2}}=\left(\frac{1^{\frac{1}{2}}}{(1-v^{2})^{\frac{1}{2}}}\right)=\frac{1}{\sqrt{1-v^{2}}}$$

$$\sinh(\phi)=\sinh(\tanh^{-1}v)=\sinh\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)$$

$$\frac{e^{\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)}-e^{\left(-\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)}}{2}=\frac{e^{\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)}-\frac{1}{e^{\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)}}}{2}=\frac{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}-\frac{1}{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}}{2}=$$

$$\frac{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}\frac{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}-\frac{1}{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}}{2}=\frac{\frac{\left(\frac{1+v}{1-v}\right)-1}{\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}}{2}=\frac{\frac{1+v}{1-v}-1}{2\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}=\frac{\frac{1+v}{1-v}-\frac{1-v}{1-v}}{2\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}=\frac{\frac{2v}{1-v}}{2\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}=$$

$$\frac{\frac{v}{1-v}\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}{1\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}=\frac{\frac{v}{1-v}\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}{\left(\frac{1+v}{1-v}\right)}=\frac{\frac{v}{1-v}\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}(1-v)}{1+v}=\frac{v\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}}{1+v}=v\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}\frac{1}{1+v}=$$

$$v\left(\frac{1+v}{1-v}\left(\frac{1}{1+v}\right)\left(\frac{1}{1+v}\right)\right)^{\frac{1}{2}}=v\left(\frac{1}{1-v}\left(\frac{1}{1+v}\right)\right)^{\frac{1}{2}}=v\left(\frac{1}{1-v^2}\right)^{\frac{1}{2}}=v\left(\frac{1^{\frac{1}{2}}}{(1-v^2)^{\frac{1}{2}}}\right)=\frac{v}{\sqrt{1-v^2}}$$

so

$$\cosh\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)=\frac{1}{\sqrt{1-v^2}}=\gamma$$

and

$$\sinh\left(\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\right)=\frac{v}{\sqrt{1-v^2}}=v\gamma$$

and using these results in the original transformation equations,

$$t^{'}=t\cosh(\phi)-x\sinh(\phi)$$
$$x^{'}=-t\sinh(\phi)+x\cosh(\phi)$$

we obtain the Lorentz equations:

$$t^{'}=t\gamma-xv\gamma=\gamma(t-xv)$$
$$x^{'}=-tv\lambda+x\gamma=\gamma(x-vt)$$

Where

$$\gamma=\frac{1}{\sqrt{1-v^2}}$$