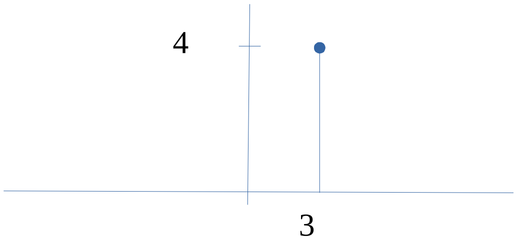
What is the derivative of this function at $x=3$?



I.e. $f_0(x) = 4$ if $x = 3$, 0 otherwise.

Recall the definition of “limit”:

For all $\epsilon > 0$ there exists some $\delta > 0$ such that the following hold

a. $0 < |x - c| < \delta \Rightarrow -(x - c) < \delta < x - c$

b. $|f(x) - L| < \epsilon$ for all x that satisfy a.

$$\lim_{x \to c} f(x) = L$$

Definition of the derivative of a function

$$\frac{df(x)}{dy} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\Delta x = |x - c|$$

So, for this case

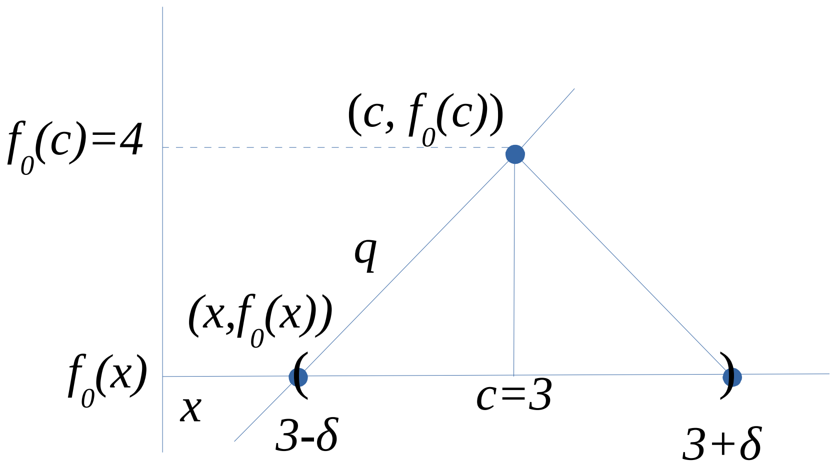
$$f(x) = \frac{f_0(x + \Delta x) - f_0(x)}{\Delta x}$$

Since $c = x + \Delta x$ and $\Delta x = c - x$

$$f(x) = \frac{f_0(c) - f_0(x)}{c - x}$$

so,

$$\frac{df(c)}{dx} = \lim_{x \to c} \frac{f_0(c) - f_0(x)}{c - x}$$



A table of the relevant terms involved and their values:

x	*c*	*f(x)*	*f₀(c)*	*f₀(x)*	*c-x*	δ	comment
1	3	4	4	0	2	2.1	3-δ<x<c
2.5	3	8	4	0	0.5	2.1	"
2.999	3	4000	4	0	0.001	2.1	"
2.9...	3	<∞	4	0	0.0...	1.1	"
3	3	∞	4	0	0	2.1	*x=c*
4	3	-4	4	0	-1	2.1	c<x<3+δ
3.1	3	-40	4	0	-0.1	2.1	"
3.0001	3	-40000	4	0	-0.0001	2.1	"
3.0...1	3	>-∞	4	0	-0.0...1	2.1	"

x approaches c from below

- As $x \to c$, $3-\delta<x<c$, $c-x>0$
  - $(c-x) \to 0 \Rightarrow f(x) \to +\infty$

x approaches c from above

- As $x \to c$, $c<x<3+\delta$, $c-x<0$
  - $(c-x) \to 0 \Rightarrow f(x) \to -\infty$

So for every $\varepsilon > 0$ there is no δ such that $|f(x) - L|>\varepsilon$ for any and all x in D and any L, because if $x<c$, $f(x)>0$ (positive) and if $x<c$, $f(x)<0$ (negative). Therefore

$$\lim_{x \to 3} f(x)$$

Does not exist.

Note:

$f: D \to \mathbb{R}$ is a function defined on a subset $D \subseteq \mathbb{R}$

$D = \{x|0<|x-c|< \delta\} \Rightarrow \{x|\ c-\delta<x<c+\delta\}$

Now, when x approaches c from below, the limit

$$\lim_{x \to 3, x<3} f(x) = +\infty$$

does exist. This is called the left derivative.

When x approaches c from above, the limit

$$\lim_{x \to 3, x > 3} f(x) = -\infty$$

also exists. This is called the right derivative.