A detailed derivation of a linear, discrete-time, Wiener Filter

Sequence *u(0), u(1), u(2),…* is the input to a filter whose filter coefficients, $w_1, w_2, w_3, \ldots = w_n$, is the impulse response of the filter. Both the input sequence terms and the impulse response terms are complex.

Note: *u(n)* is <u>*NOT*</u> the unit step function!

$$y(n) = \sum_{k=0}^{\infty} w_k^* u(n-k), n = 0, 1, 2, \ldots$$

* = complex conjugation

$w_k^* u(n-k)$ represents an inner product of the filter coefficients w_k and the filter input *u(n-k).*

d(n) = the desired sequence

We wish to generate *y(n)*, an <u>estimate of the desired sequence</u>, that is related to the desired sequence with the following expression:

$$e(n) = d(n) - y(n)$$

The filter design is optimized if the mean-square error (MSE) value of the estimation error, *e(n)* is minimized.

We define

$$J = E[e(n)e^*(n)]$$
$$= E\left[||e(n)||^2\right]$$

Since the coefficients are complex, they have a real and imaginary part:

$$w_k = a_k + jb_k, \text{k} = 0,1,2, \ldots$$

We take the derivative of the filter expression and set this equal to 0, then solve for the values of the coefficients. This is modified version of the technique used to find the minimum of a continuous function.

We use the gradient operator ∇, the kth element of which is written in terms of the first-order partial derivatives with respect to the real part a_k and the imaginary part b_k, for the *k*th filter coefficient:

$$\nabla_k = \frac{\partial}{\partial a_k} + j\frac{\partial}{\partial b_k}, k = 0,1,2, \ldots$$

$$\nabla_k(J) = \frac{\partial Re[J]}{\partial a_k} + j\frac{\partial Imj[J]}{\partial b_k}, k = 0,1,2, \ldots$$

For the cost function, J, to reach its minimum value, all the elements of the gradient vector $\nabla(J)$ must be simultaneously equal to zero, as shown by

$$\nabla_k = 0, k = 0,1,2, \ldots$$

$$\nabla_k(J) = \nabla_k([e(n)e^*(n)])$$

The expected value of two random processes, r and s:

$$\mathbb{E}[r(n)s(n)]$$

$$\mathbb{E}[r(n)] = r(0)P_{r0} + r(1)P_{r1} + \cdots$$

$$\mathbb{E}[s(n)] = s(0)P_{s0} + s(1)P_{s1} + \cdots$$

$$\mathbb{E}[r(n)s(n)] = r(0)P_{r0}s(0)P_{s0} + r(1)s(1)P_{s1} + \cdots$$

Note: This is *not* the product of two polynomials, but the statistical inner product (SIP) of two sequences.

The partial derivative of the MSE equals the MSE of the partial derivative of the SIP of r(n) and s(n)

$$\frac{\partial \mathbb{E}[r(n)s(n)]}{\partial q} = \frac{\partial[r(0)P_{r0}s(0)P_{s0} + r(1)P_{r1}s(1)P_{s1} + \cdots]}{\partial q}$$

$$= \frac{\partial(r(0)P_{r0}s(0)P_{s0}) + \partial(r(1)P_{r1}s(1)P_{s1}) + \cdots)}{\partial q}$$

$$\mathbb{E}\left[\frac{\partial(r(n)P_{rn}s(n)P_{sn})}{\partial q}\right]$$

For $E[e(n)e^*(n)] = J$

$$\mathbb{E}[e(n)e^*(n)] = \mathbb{E}[\big(d(n) - y(n)\big)\big(d(n) - y^*(n)\big)]$$

$$= \big(d(0) - y(0)\big)P_{(d-y)0}\big(d(0) - y^*(0)\big)P_{(d-y)0}$$

$$+\big(d(1) - y(1)\big)P_{(d-y)1}\big(d(1) - y^*(1)\big)P_{(d-y)1}$$

$$+\big(d(2) - y(2)\big)P_{(d-y)2}\big(d(2) - y^*(2)\big)P_{(d-y)2}$$

$$+ \cdots]$$

$$= \big(d(0) - y(0)\big)\big(d(0) - y^*(0)\big)P_{(d-y)0}P_{(d-y)0}$$

$$+\big(d(1) - y(1)\big)\big(d(1) - y^*(1)\big)P_{(d-y)1}P_{(d-y)1}$$

$$+\left(d(2)-y(2)\right)\left(d(2)-y^*(2)\right)P_{(d-y)2}P_{(d-y)2}+\cdots]$$

$$\frac{\partial((d(0)-y(0))\left(d(0)-y^*(0)\right)P_{(d-y)0}{}^2}{\partial a_0}+\frac{\partial((d(0)-y(0))\left(d(0)-y^*(0)\right)P_{(d-y)0}{}^2}{\partial b_0}$$

$$\frac{\partial((d(1)-y(1))\left(d(1)-y^*(1)\right)P_{(d-y)1}{}^2}{\partial a_0}+\frac{\partial((d(1)-y(1))\left(d(1)-y^*(1)\right)P_{(d-y)1}{}^2}{\partial b_0}$$

$$+\cdots$$

The partial derivative of the two sequences uses the familiar formula of the derivative of the product of two functions from calculus:

$$\frac{\partial a(n)b(n)}{\partial q}=\frac{\partial a(n)}{\partial q}*\partial b(n)+\frac{\partial b(n)}{\partial q}*\partial a(n)$$

So, since

$$y(n)=\sum_{k=0}^{\infty}w_k^*\,u(n-k), n=0,1,2,\ldots$$

$$w_k = a_k + jb_k,\ \mathrm{k} = 0,1,2,\ \ldots$$

$$\frac{\partial((d(n)-y(n))\left(d(n)-y^*(n)\right)P_{(d-y)n}{}^2}{\partial a_0}$$

$$=P_{(d-y)n}{}^2\left(\frac{\partial\left(d(n)-y(n)\right)}{\partial a_n}\right)\left(d(n)-y^*(n)\right)+\left(\frac{\partial\left(d(n)-y^*(n)\right)}{\partial a_n}\right)\left(d(n)-y(n)\right)$$

$$\frac{\partial(d(n)-y(n))}{\partial a_k}$$

$$=\frac{\partial d(n)}{\partial a_k}-\frac{\partial y(n)}{\partial a_k}$$

Since *d(n)* does not depend on a_k. or b_k, $\frac{\partial d(n)}{\partial a_k}=0$

$$=0-\frac{\partial y(n)}{\partial a_k}=u(n-k)$$

$$0-u(n-k))e(n)^*+0-u(n-k)e(n)$$

and

$$P_{(d-y)n}{}^2\left(\frac{j\partial\left(d(n)-y(n)\right)}{\partial b_n}\right)\left(d(n)-y^*(n)\right)+\left(\frac{j\partial\left(d(n)-y^*(n)\right)}{\partial b_n}\right)\left(d(n)-y(n)\right)$$

$$(0 - j * ju(n-k))e(n)^*) + \left(0 - j * ju(n-k)\right)e(n)$$

$$u(n-k)e(n)^* + u(n-k)e(n)$$

Summing the two results

$$u(n-k))e(n)^* - u(n-k)e(n) + u(n-k)e(n)^* + u(n-k)e(n)$$

$$= 2u(n-k))e(n)^*$$

$$\Rightarrow \nabla_k(J) = \mathbb{E}\left[2u(n-k)\right)e(n)^*]$$

"Let e_0 denote the special value of the estimation error that results when the filter operates in its optimum condition, so the filter is operating in its optimal state when the following is true, or as close to being true as is possible

$$\mathbb{E}\left[u(n-k)\right)e_0(n)^*] = 0$$

In other words $\mathbb{E}\left[u(n-k)\right)e_0(n)^*]$ is as close to 0 as possible.